Locally Finite ω-Languages and Effective Analytic Sets Have the Same Topological Complexity

Olivier Finkel

Locally Finite ω-Languages and Effective Analytic Sets Have the Same Topological Complexity

Olivier Finkel
Equipe de Logique Mathématique
Institut de Mathématiques de Jussieu - Paris Rive Gauche
CNRS et Université Paris Diderot Paris 7
Bâtiment Sophie Germain Case 7012
75205 Paris Cedex 13, France.
finkel@math.univ-paris-diderot.fr

Abstract

Local sentences and the formal languages they define were introduced by Ressayre in [Res88]. We prove that locally finite ω-languages and effective analytic sets have the same topological complexity: the Borel and Wadge hierarchies of the class of locally finite ω-languages are equal to the Borel and Wadge hierarchies of the class of effective analytic sets. In particular, for each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete locally finite ω-languages, and the supremum of the set of Borel ranks of locally finite ω-languages is the ordinal γ_2^1, which is strictly greater than the first non-recursive ordinal ω_1^{CK}. This gives an answer to the question of the topological complexity of locally finite ω-languages, which was asked by Simonnet [Sim92] and also by Duparc, Finkel, and Ressayre in [DFR01]. Moreover we show that the topological complexity of a locally finite ω-language defined by a local sentence φ may depend on the models of the Zermelo-Fraenkel axiomatic system **ZFC**. Using similar constructions as in the proof of the above results we also show that the equivalence, the inclusion, and the universality problems for locally finite ω-languages are Π^1_2-complete, hence highly undecidable.

Keywords: Local sentences; logic in computer science; formal languages; locally finite languages; infinite words; ω-languages; topological complexity; Borel hierarchy; Wadge hierarchy; models of set theory; decision problems.

1 Introduction

Local sentences were introduced by Ressayre who proved in [Res88] some remarkable stretching theorems which established some links between the finite and the infinite model theory of these sentences. These theorems show that the existence of some well ordered models of a local sentence φ (a binary relation symbol is here assumed to belong to the signature of φ and to be interpreted by a linear order in every model of φ) is equivalent to the existence of some finite model of φ, generated by some particular kind of indiscernibles, like special, remarkable or monotonic ones. In particular, a local sentence φ has a model of order type ω if and only if it has a finite model generated by N_φ special indiscernibles (where N_φ is a positive integer depending on φ), and a similar result establishes a connection between the existence of a model of order type α (where α is an ordinal $< \omega^\omega$) and the existence of a finite model (of another local sentence φ_α) generated by semi-monotonic indiscernibles [FR96].

These theorems provide some decision algorithms which show the decidability of the following problem: "For a given local sentence φ and an ordinal $\alpha < \omega^\omega$, has φ a model of order type α ?"

These results look like Büchi's one about the decidability of the monadic second order theory of one successor over the integers [Büc62], and even more like its extension: the decidability of the monadic second order theory of the structure $(\alpha, <)$ for a countable ordinal α. In order to prove this result, Büchi studied in the sixties the class of ω-languages accepted by finite automata with what is now called Büchi acceptance condition. He showed that an ω-language, i.e. a set of words of length ω over a finite alphabet, is accepted by a finite automaton with the Büchi acceptance condition if and only if it is defined by a monadic second order sentence [Büc62, Tho90]. The equivalence between definability by monadic second order sentences and acceptance by finite automata, which is also true for languages of finite words, has then been extended to α-languages, i.e. languages of words of length α, where α is a countable ordinal $\geq \omega$ [BS73]. This led to decision algorithms showing that the monadic second order theory of the structure $(\alpha, <)$ is decidable.

A way to compare the power of the above decidability results concerning local or monadic sentences is to compare the expressive power of monadic sentences and of local sentences, and then to consider languages defined by these sentences. Ressayre introduced locally finite languages which are defined by local sentences. Local sentences are first order, but they define locally finite languages via existential quantifications over relations and functions which appear in the local sentence. These second order quantifications are more general than the monadic ones. When finite words are considered, each regular language is locally finite, [Res88], and many context-free as well as non-context-free languages are locally finite [Fin01]. Moreover it was proved in [Fin89, Fin01] that the class LOC_α of locally finite α-languages, for $\omega \leq \alpha < \omega^\omega$, is a strict extension of the class REG_α of regular α-languages (defined by monadic second order sentences). Then the following question naturally arises:

"How large is the extension of REG_α by LOC_α ?"

We have begun to attack this problem by comparing the topological complexity of ω-languages in each of these classes, and firstly to locate them with regard to the Borel and projective hierarchies. On one side it is well known that all ω-regular languages are boolean combinations of $\mathbf{\Sigma}^0_2$-sets hence $\mathbf{\Delta}^0_3$-sets, [Tho90, PP04]. On the other side we proved in [Fin08] that locally finite ω-languages extend far beyond regular ω-languages: the class LOC_ω meets all finite levels of the Borel hierarchy, contains some Borel sets of infinite rank and even some analytic but non-Borel sets.

Notice that the question of the topological complexity of locally finite ω-languages is also motivated by the general project of studying the logical definability of classes of formal languages of (finite or) infinite words, (or of relational structures like graphs); see [Pin96, Tho97] for a survey about this field of research called "descriptive complexity".

In [Fin06] we proved that the Borel and Wadge hierarchies of the class of ω-languages accepted by real-time 1-counter Büchi automata are equal to the Borel and Wadge hierarchies of the class of effective analytic sets accepted by Büchi Turing machines.

Using this previous result, we prove in this paper that locally finite ω-languages and effective analytic sets have the same topological complexity: the Borel and Wadge hierarchies of the class of locally finite ω-languages are equal to the Borel and Wadge hierarchies of the class of effective analytic sets. In particular, for each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete locally finite ω-languages, and the supremum of the set of Borel ranks of locally finite ω-languages is the ordinal γ_2^1, (see [KMS89] for more precision), which is strictly greater than the first non-recursive ordinal ω_1^{CK}.

This gives an answer to the question of the topological complexity of locally finite ω-languages, which was asked by Simonnet [Sim92] and also by Duparc, Finkel, and Ressayre in [DFR01].

Moreover we show that the topological complexity of a locally finite ω-language defined by a local sentence φ may depend on the models of the Zermelo-Fraenkel axiomatic system **ZFC**.

Using similar constructions as in the proof of the above results, we also show that the equivalence, the inclusion, and the universality problems for locally finite ω-languages are Π^1_2-complete, hence highly undecidable.

The paper is organized as follows. In section 2 we review the definitions and some properties of local sentences and locally finite (omega) languages. Then we give some examples of locally finite ω-languages. In section 3 we recall notions of topology, and in particular the Borel and Wadge hierarchies on a Cantor space. In section 4 we study topological properties of locally finite ω-languages.

2 Review of local sentences and languages

2.1 Definitions and properties of local sentences

In this paper the (first order) signatures are finite, always contain one binary predicate symbol $=$ for equality, and can contain both functional and relational symbols. The terms, open formulas and formulas are built in the usual way.

When M is a structure in a signature Λ and $X \subseteq |M|$, we define:
$cl^1(X, M) = X \cup \bigcup_{\{f \text{ n-ary function of } \Lambda\}} f^M(X^n) \cup \bigcup_{\{a \text{ constant of } \Lambda\}} a^M$
$cl^{n+1}(X, M) = cl^1(cl^n(X, M), M) \quad$ for an integer $n \geq 1$
and $cl(X, M) = \bigcup_{n \geq 1} cl^n(X, M)$ is the closure of X in M.

Let us now define local sentences. We shall denote S(φ) the signature of a first order sentence φ, i.e. the set of non-logical symbols appearing in φ.

Definition 2.1 A first order sentence φ is local if and only if:

a) $M \models \varphi$ and $X \subseteq |M|$ imply $cl(X, M) \models \varphi$

b) $\exists n \in \mathbb{N}$ such that $\forall M$, if $M \models \varphi$ and $X \subseteq |M|$, then $cl(X, M) = cl^n(X, M)$, (closure in models of φ takes at most n steps).

Notation. For a local sentence φ, let n_φ be the smallest integer $n \geq 1$ verifying b) of the above definition.

Remark 2.2 Because of a) of Definition 2.1, a local sentence φ is always equivalent to a universal sentence, so we may assume that φ is universal.

Let us now state first properties of local sentences.

Theorem 2.3 (Ressayre, see [Fin01])

(a) The set of local sentences is recursively enumerable.

(b) It is undecidable whether an arbitrary sentence φ is a local one.

Per contra to these negative results, there exists a "recursive presentation" up to logical equivalence of all local sentences.

Theorem 2.4 (Ressayre, see [Fin01]) *There exist a recursive set* $\mathbf{L}$ *of local sentences and a recursive function* $\mathbf{F}$ *such that:*

1) ψ local $\longleftrightarrow \exists \psi' \in \mathbf{L}$ such that $\psi \equiv \psi'$.

2) $\psi' \in \mathbf{L} \longrightarrow n_{\psi'} = \mathbf{F}(\psi')$.

The elements of $\mathbf{L}$ are the $\psi \wedge C_n$, where ψ run over the universal formulas and C_n run over the universal formulas in the signature $S(\psi)$ which express that closure in a model takes at most n steps.
$\psi \wedge C_n$ is local and $n_{\psi \wedge C_n} \leq n$. Then we can compute $n_{\psi \wedge C_n}$, considering only finite models of cardinal $\leq m$, where m is an integer depending on n. And each local sentence ψ is equivalent to a universal formula θ, hence $\psi \equiv \theta \wedge C_{n_\psi}$.

We shall restrict in the sequel our attention to local sentences with a binary predicate $<$ in their signature which is interpreted by a linear ordering in all of their models.

A fundamental result about local sentences is the stretching theorem, see [FR96] which shows the existence of remarkable connections between the finite and the infinite model theory of local sentences. The stretching theorem implies the existence of decision procedures for several problems. Notice that the set of local sentences is not recursive but we can consider that the algorithms given by the following theorem are applied to local sentences in the recursive set $\mathbf{L}$ given by Proposition 2.4. In particular φ is given with the integer n_φ.

Theorem 2.5 ([FR96]) *It is decidable, for a given local sentence φ, whether*

(1) φ has arbitrarily large finite models.

(2) φ has an infinite model.

(3) φ has an infinite well ordered model.

(4) φ has a model of order type ω.

(5) φ has well ordered models of unbounded order types in the ordinals.

On the other side Büchi showed that one can decide whether a monadic second order formula of $S1S$ is true in the structure $(\omega, <)$. But for a formula of size n his procedure might run in time

$$\underbrace{2^{2^{\cdot^{\cdot^{2^n}}}}}_{\mathbf{O(n)}}$$

see [Büc62] for more details. Moreover it has been proved by Meyer that one cannot essentially improve this result: the monadic second order theory of the structure $(\omega, <)$ is not elementary recursive, [Mey75]. Notice that the complexity of Büchi's algorithm for monadic sentences is in terms of the length of the formula and the complexity of the algorithms for local sentences is in terms of the length of a local sentence φ *and* the corresponding integer N_φ. But, as explained in [Fin08], the algorithms for local sentences given by Theorem 2.5 are of much lower complexity than the algorithm for decidability of $S1S$. This is remarkable because the expressive power of local sentences is also greater than the expressive power of monadic second order sentences, as shown in [Fin08] and in this paper.

2.2 Definitions and first properties of local languages

Let us now introduce notations for words. Let Σ be a finite alphabet whose elements are called letters. A finite non-empty word over Σ is a finite sequence of letters: $x = a_1 a_2 \cdots a_n$ where $\forall i \in [1; n]$ $a_i \in \Sigma$. We shall denote $x(i) = a_i$ the i^{th} letter of x and $x[i] = x(1) \cdots x(i)$ for $i \leq n$. The length of x is $|x| = n$. The empty word will be denoted by λ and has 0 letters. Its length is 0. The set of finite words over Σ is denoted $\Sigma^\star$. $\Sigma^+ = \Sigma^\star - \{\lambda\}$ is the set of non-empty words over Σ. A (finitary) language L over Σ is a subset of $\Sigma^\star$. Its complement (in $\Sigma^\star$) is $L^- = \Sigma^\star - L$. The usual concatenation product of u and v will be denoted by $u \cdot v$ or just uv. For $V \subseteq \Sigma^\star$, we denote $V^\star = \{v_1 \cdots v_n \mid n \in \mathbb{N} \quad and \quad \forall i \in [1; n] \quad v_i \in V\}$.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 a_2 \cdots a_n \cdots$, where $\forall i \geq 1$ $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \cdots$ and $\sigma[n] = \sigma(1)\sigma(2) \cdots \sigma(n)$ the finite word of length n, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \cdots u_n \cdots \in \Sigma^\omega \mid \forall i \geq 1$ $u_i \in V\}$ is the ω-power of V. For a subset $A \subseteq \Sigma^\omega$, the complement of A (in Σ^ω) is $\Sigma^\omega - A$ denoted A^-. The concatenation product is extended to the product of a finite word u and an ω-word v: the infinite word $u \cdot v$ is then the ω-word such that: $(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$.

The prefix relation is denoted $\sqsubseteq$: the finite word u is a prefix of the finite word v (respectively, the infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u \cdot w$.

A word over Σ may be considered as a structure in the following usual manner: Let Σ be a finite alphabet. We denote P_a a unary predicate for each letter $a \in \Sigma$ and Λ_Σ the signature $\{<, (P_a)_{a \in \Sigma}\}$. Let σ be a finite word over the alphabet Σ, $|\sigma|$ is the length of the word σ. We may write that $|\sigma| = \{1, 2, \ldots, |\sigma|\}$. σ is identified with the structure $(|\sigma|, <^\sigma, (P_a^\sigma)_{a \in \Sigma})$ of signature Λ_Σ where
$P_a^\sigma = \{1 \leq i \leq |\sigma| \mid$ the i^{th} letter of σ is an $a\}$.
In a similar manner if σ is an ω-word over the alphabet Σ, then ω is the length of the word σ and we may write $|\sigma| = \{1, 2, 3, \ldots\}$. σ is identified to the structure $(|\sigma|, <^\sigma, (P_a^\sigma)_{a \in \Sigma})$ of signature Λ_Σ where
$P_a^\sigma = \{1 \leq i < \omega \mid$ the i^{th} letter of σ is an $a\}$.

Definition 2.6 Let Σ be a finite alphabet and $L \subseteq \Sigma^\star$ be a language of finite words (respectively, $L \subseteq \Sigma^\omega$ be a language of infinite words) over the alphabet Σ. Then L is a locally finite language (respectively, ω-language) $\longleftrightarrow$ there exists a local sentence φ in a signature $\Lambda \supseteq \Lambda_\Sigma$ such that $\sigma \in L$ iff $\exists$ finite M, (respectively, $\exists M$ of order type ω) $M \models \varphi$ and $M|\Lambda_\Sigma = \sigma$ (where $M|\Lambda_\Sigma$ is the reduction of M to the signature Λ_Σ).
We then denote $L = L^\Sigma(\varphi)$ (respectively, $L = L^\Sigma_\omega(\varphi)$), and to simplify, when there is no ambiguity, $L = L(\varphi)$ (respectively, $L = L_\omega(\varphi)$) the locally finite language (respectively, ω-language) defined by φ.
The class of locally finite languages will be denoted LOC.
The class of locally finite ω-languages will be denoted LOC_ω.

The empty word λ has 0 letters. It is represented by the empty structure. Recall that if $L(\varphi)$ is a locally finite language then $L(\varphi) - \{\lambda\}$ and $L(\varphi) \cup \{\lambda\}$ are also locally finite [Fin01].

Remark 2.7 The notion of locally finite language is very different from the usual notion of local language which represents a subclass of the class of rational languages. But from now on, as in [Fin01], things being well defined and made precise, we shall call simply local languages the locally finite languages.

Let us state the following decidability results.

Theorem 2.8 *It is decidable, for a local sentence φ, given with the integer n_φ, and an alphabet Σ, whether*

(1) The local language $L^\Sigma(\varphi)$ is empty.

(2) The local language $L^\Sigma(\varphi)$ is infinite.

(3) The local ω-language $L^\Sigma_\omega(\varphi)$ is empty.

Remark 2.9 *Item (3) states that the emptiness problem for local ω-languages is decidable. It relies on a remarkable analogue to the property: "a regular ω-language is non-empty iff it contains an ultimately periodic word, i.e. an ω-word in the form $u \cdot v^\omega$ where u and v are finite words".*
When local ω-languages are considered, this property becomes: "a local ω-language is non-empty iff it contains an ω-word which is the reduction, to the signature of words, of an ω-model generated by special indiscernibles", see [Res88, FR96].

2.3 Examples of local languages

We first recall the following result which shows that the class of local (finitary) languages extends the class of regular ones.

Theorem 2.10 (Ressayre, see [Res88]) *The class of local (finitary) languages is closed under finite union, concatenation product, and star operation. This implies that each regular (finitary) language is local.*

We now give some examples of local ω-languages.

Example 2.11 ([Fin04]) The ω-language which contains only the word $\sigma = abab^2ab^3ab^4\ldots$ is a local ω-language over the alphabet $\{a, b\}$. *Notice that this ω-language is not regular since its single ω-word is not ultimately periodic.*

Recall that for any family $\mathcal{L}$ of finitary languages, the ω-Kleene closure of $\mathcal{L}$, is:

$$\omega{-}KC(\mathcal{L}) = \{ \bigcup_{1 \leq i \leq n} U_i \cdot V_i^\omega \mid \forall i \in [1, n]\ U_i, V_i \in \mathcal{L}\}$$

It is well known that the class REG_ω of regular ω-languages (respectively, the class CF_ω of context free ω-languages) is the ω-Kleene closure of the family REG of regular finitary languages (respectively, of the family CF of context free finitary languages), [Tho90, Sta97].

We showed that a similar characterization does not hold for local languages.

Theorem 2.12 ([Fin04]) *The ω-Kleene closure of the class LOC of finitary local languages is strictly included in the class LOC_ω of local ω-languages.*

Then we easily derive the following example because every regular finitary language is local [Res88].

Example 2.13 ([Fin01]) Every regular ω-language is a local ω-language, i.e. $REG_\omega \subseteq LOC_\omega$.

Since numerous context free languages are local [Fin01], $CF_\omega = \omega{-}KC(CF)$ implies that many context free ω-languages are local. The problem whether every context free ω-language is local is still open but by Theorem 2.12, $CF \subseteq LOC$ would imply that $CF_\omega \subseteq LOC_\omega$.

Example 2.14 The ω-languages U^ω and $U.a^\omega$, where $U = \{a^{n^2}b^{n^2}c^{n^2} \mid n \geq 1\}$ is a local finitary language over the alphabet $\{a, b, c\}$, are examples of local but non context free ω-languages, [Fin01].

Recall that a substitution f is defined by a mapping $\Sigma \rightarrow P(\Gamma^\star)$, where $\Sigma = \{a_1, ..., a_n\}$ and Γ are two finite alphabets, $f : a_i \rightarrow L_i$ where $\forall i \in [1; n]$, L_i is a finitary language over the alphabet Γ. The substitution is said to be λ-free if $\forall i \in [1; n]$, L_i does not contain the empty word λ. It is a (λ-free) morphism when every language L_i contains only one (non-empty) word. Now this mapping is extended in the usual manner to finite words and to finitary languages: for some letters $x(1), x(2), \ldots, x(n)$ in Σ, $f(x(1)x(2)\cdots x(n)) = \{u_1u_2\cdots u_n \mid \forall i \in [0; n] \;\; u_i \in f(x(i))\}$, and for $L \subseteq \Sigma^\star$, $f(L) = \cup_{x\in L} f(x)$. If the substitution f is λ-free, we can extend this to ω-words and ω-languages: $f(x(1)x(2)\cdots x(n)\cdots) = \{u_1u_2\cdots u_n\cdots \mid \forall i \geq 0 \;\; u_i \in f(x(i))\}$ and for $L \subseteq \Sigma^\omega$, $f(L) = \cup_{x\in L} f(x)$.

We now recall some closure properties of the class LOC_ω which allow us to generate many other local ω-languages from the known ones.

Theorem 2.15 ([Fin04]) *The class LOC_ω is closed under union, left concatenation with local (finitary) languages, λ-free substitution of local (finitary) languages, λ-free morphism.*

3 Topology

3.1 Borel hierarchy and analytic sets

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n + 1)^{st}$ letter of u is different from the $(n + 1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W \cdot \Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

Define now the *Borel Hierarchy* of subsets of Σ^ω:

Definition 3.1 *For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:*
$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of Σ^ω,
and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

The class of *Borel sets* is $\mathbf{\Delta}^1_1 := \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi$, where ω_1 is the first uncountable ordinal. There are also some subsets of Σ^ω which are not Borel. In particular the class of Borel subsets of Σ^ω is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets.

Definition 3.2 *A subset A of Σ^ω is in the class $\mathbf{\Sigma}_1^1$ of analytic sets iff there exists another finite set Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where (x, y) is the infinite word over the alphabet $\Sigma \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.*

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}_\alpha^0$ (respectively, $\mathbf{\Pi}_\alpha^0$, $\mathbf{\Sigma}_1^1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}_\alpha^0$ (respectively, $E \in \mathbf{\Pi}_\alpha^0$, $E \in \mathbf{\Sigma}_1^1$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$.

Let us now recall the definition of the arithmetical hierarchy of ω-languages, see for example [Sta97, Mos80]. Let Σ be a finite alphabet. An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times \Sigma^\star$ such that

$$L = \{\sigma \in \Sigma^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Π_n if and only if its complement $\Sigma^\omega - L$ belongs to the class Σ_n. The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy and the classes Σ_n and Π_n are strictly included in the respective classes $\mathbf{\Sigma}_n^0$ and $\mathbf{\Pi}_n^0$ of the Borel hierarchy.

As in the case of the Borel hierarchy, projections of arithmetical sets (of the second Π-class) lead beyond the arithmetical hierarchy, to the analytical hierarchy of ω-languages. The first class of the analytical hierarchy of ω-languages is the (lightface) class Σ_1^1 of effective analytic sets. An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Σ_1^1 if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N}) \times \{0,1\}^\star \times \Sigma^\star$ such that:

$$L = \{\sigma \in \Sigma^\omega \mid \exists \tau (\tau \in \{0,1\}^\omega \wedge \forall n \exists m ((n, \tau[m], \sigma[m]) \in R_L))\}$$

Thus an ω-language $L \subseteq \Sigma^\omega$ is in the class Σ_1^1 iff it is the projection of an ω-language over the alphabet $\{0,1\} \times \Sigma$ which is in the class Π_2 of the arithmetical hierarchy.

We recall the following result which gives a first upper bound on the complexity of local ω-languages.

Theorem 3.3 ([Fin08]) *The class LOC_ω is strictly included in the class Σ_1^1.*

By Suslin's Theorem [Kec95, page 226], an analytic subset of Σ^ω is either countable or has the continuum power. Then we can infer the following:

Corollary 3.4 *Let Σ be a finite alphabet. Every local ω-language $L_\omega^\Sigma(\varphi)$ over the alphabet Σ is either countable or has the continuum power.*

Kechris, Marker and Sami proved in [KMS89] that the supremum of the set of Borel ranks of (lightface) Π_1^1, so also of (lightface) Σ_1^1, sets is the ordinal γ_2^1. This ordinal is precisely defined in [KMS89]. Kechris, Marker and Sami proved that the ordinal γ_2^1 is strictly greater than the ordinal δ_2^1 which is the first non Δ_2^1 ordinal. Thus in particular it holds that $\omega_1^{\mathrm{CK}} < \gamma_2^1$, where ω_1^{CK} is the first non-recursive ordinal, usually called the Chruch-Kleene ordinal. The exact value of the ordinal γ_2^1 may depend on axioms of set theory [KMS89]. It is consistent with the axiomatic system **ZFC** that γ_2^1 is equal to the ordinal δ_3^1 which is the first non Δ_3^1 ordinal (because $\gamma_2^1 = \delta_3^1$ in **ZFC + (V=L)**). On the other hand the axiom of Π_1^1-determinacy implies that $\gamma_2^1 < \delta_3^1$. For more details, the reader is referred to [KMS89] and to a textbook of set theory like [Jec02].

Notice however that it seems still unknown whether *every* non null ordinal $\gamma < \gamma_2^1$ is the Borel rank of a (lightface) Π_1^1 (or Σ_1^1) set. On the other hand it is known that every ordinal $\gamma < \omega_1^{\mathrm{CK}}$ is the Borel rank of a (lightface) Δ_1^1-set, since for every ordinal $\gamma < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}_\gamma^0$-complete and $\mathbf{\Pi}_\gamma^0$-complete sets in the class Δ_1^1.

3.2 Wadge hierarchy

We now introduce the Wadge hierarchy, which is a great refinement of the Borel hierarchy defined via reductions by continuous functions, [Dup01, Wad83].

Definition 3.5 (Wadge [Wad83]) *Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.*
L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.
A set $L \subseteq X^\omega$ is said to be self dual iff $L \equiv_W L^-$, and otherwise it is said to be non self dual.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The *equivalence classes* of $\equiv_W$ are called *Wadge degrees*.
The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, if $L \leq_W L'$ and $L = f^{-1}(L')$ where f is a continuous function from X^ω into Y^ω, then f is called a continuous reduction of L to L'. Intuitively it means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function. Hence the Wadge degree of an ω-language is a measure of its topological complexity.
Notice that in the above definition, we consider that a subset $L \subseteq X^\omega$ is given together with the alphabet X.
We can now define the *Wadge class* of a set L:

Definition 3.6 *Let L be a subset of X^ω. The Wadge class of L is :*

$$[L] = \{L' \mid L' \subseteq Y^\omega \textit{ for a finite alphabet } Y \textit{ and } L' \leq_W L\}.$$

Recall that each *Borel class* $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ is a *Wadge class*. A set $L \subseteq X^\omega$ is a $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$)-*complete set* iff for any set $L' \subseteq Y^\omega$, L' is in $\mathbf{\Sigma}^0_\alpha$ (respectively $\mathbf{\Pi}^0_\alpha$) iff $L' \leq_W L$.

There is a close relationship between Wadge reducibility and games which we now introduce.

Definition 3.7 *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, player 1 who is in charge of L and player 2 who is in charge of L'.*
Player 1 first writes a letter $a_1 \in X$, then player 2 writes a letter $b_1 \in Y$, then player 1 writes a letter $a_2 \in X$, and so on.
The two players alternatively write letters a_n of X for player 1 and b_n of Y for player 2.
After ω steps, the player 1 has written an ω-word $a \in X^\omega$ and the player 2 has written an ω-word $b \in Y^\omega$.
The player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps.
The player 2 wins the play iff [$a \in L \leftrightarrow b \in L'$], i.e. iff :

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L' \text{ and } b \text{ is infinite})].$$

Recall that a strategy for player 1 is a function $\sigma : (Y \cup \{s\})^\star \to X$. And a strategy for player 2 is a function $f : X^+ \to Y \cup \{s\}$.
σ is a winning stategy for player 1 iff he always wins a play when he uses the strategy σ, i.e. when the n^{th} letter he writes is given by $a_n = \sigma(b_1 \cdots b_{n-1})$, where b_i is the letter written by player 2 at step i and $b_i = s$ if player 2 skips at step i.
A winning strategy for player 2 is defined in a similar manner.

Martin's Theorem states that every Gale-Stewart Game $G(X)$ (see [Kec95]), with X a Borel set, is determined and this implies the following :

Theorem 3.8 (Wadge) *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ be two Borel sets, where X and Y are finite alphabets. Then the Wadge game $W(L, L')$ is determined : one of the two players has a winning strategy. And $L \leq_W L'$ iff the player 2 has a winning strategy in the game $W(L, L')$.*

Theorem 3.9 (Wadge) *Up to the complement and $\equiv_W$, the class of Borel subsets of X^ω, for a finite alphabet X having at least two letters, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d_W^0 from WH onto $|WH| - \{0\}$, such that for all $L, L' \subseteq X^\omega$:*
$d_W^0 L < d_W^0 L' \leftrightarrow L <_W L'$ *and*
$d_W^0 L = d_W^0 L' \leftrightarrow [L \equiv_W L'$ *or* $L \equiv_W L'^-]$.

The Wadge hierarchy of Borel sets of **finite rank** has length ${}^1\varepsilon_0$ where ${}^1\varepsilon_0$ is the limit of the ordinals α_n defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer, ω_1 being the first non countable ordinal. Then ${}^1\varepsilon_0$ is the first fixed point of the ordinal exponentiation of base ω_1. The length of the Wadge hierarchy of Borel sets in $\mathbf{\Delta}^0_\omega = \mathbf{\Sigma}^0_\omega \cap \mathbf{\Pi}^0_\omega$ is the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1, which is a much larger ordinal. The length of the whole Wadge hierarchy of Borel sets is a huge ordinal, with regard to the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1. It is described in [Wad83, Dup01] by the use of the Veblen functions.

4 Topological complexity of local ω-languages

We now firstly recall the notion of real-time 1-counter Büchi automaton which will be useful in the sequel.

A 1-counter machine has one *counter* containing a non-negative integer. The machine can test whether the content of the counter is zero or not. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counter. Notice that in this model some λ-transitions are allowed.

Formally a 1-counter machine is a 4-tuple $\mathcal{M}$=(K, Σ, Δ, q_0), where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0,1\} \times K \times \{0,1,-1\}$ is the transition relation. The 1-counter machine $\mathcal{M}$ is said to be *real time* iff: $\Delta \subseteq K \times \Sigma \times \{0,1\} \times K \times \{0,1,-1\}$, i.e. iff there are no λ-transitions.

If the machine $\mathcal{M}$ is in state q and $c \in \mathbf{N}$ is the content of the counter then the configuration (or global state) of $\mathcal{M}$ is the pair (q, c).

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $c \in \mathbf{N}$, if $(q, a, i, q', j) \in \Delta$ where $i = 0$ if $c = 0$ and $i = 1$ if $c > 0$, then we write:

$a : (q, c) \mapsto_{\mathcal{M}} (q', c + j)$.

Thus the transition relation must obviously satisfy:
if $(q, a, i, q', j) \in \Delta$ and $i = 0$ then $j = 0$ or $j = 1$ (but j may not be equal to -1).

Let $\sigma = a_1 a_2 \cdots a_n \cdots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c_i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on σ, iff:

(1) $(q_1, c_1) = (q_0, 0)$

(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c_i) \mapsto_{\mathcal{M}} (q_{i+1}, c_{i+1})$ and such that $a_1 a_2 \cdots a_n \cdots = b_1 b_2 \cdots b_n \cdots$

For every such run r, $\mathrm{In}(r)$ is the set of all states entered infinitely often during r.

Definition 4.1 *A Büchi 1-counter automaton is a 5-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'$=(K, Σ, Δ, q_0) is a 1-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is:*
$L(\mathcal{M})$= $\{\sigma \in \Sigma^\omega \mid$ *there exists a run r of $\mathcal{M}$ on σ such that* $\mathrm{In}(r) \cap F \neq \emptyset\}$

The class of ω-languages accepted by Büchi 1-counter automata is denoted $\mathbf{BCL}(1)_\omega$. The class of ω-languages accepted by *real time* Büchi 1-counter automata will be denoted **r-**$\mathbf{BCL}(1)_\omega$.

The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class CF_ω of context free ω-languages accepted by Büchi pushdown automata.

If we omit the counter of a real-time Büchi 1-counter automaton, then we simply get the notion of Büchi automaton. The class of ω-languages accepted by Büchi automata is the class of regular ω-languages, denoted REG_ω.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ_1^1 of effective analytic sets [CG78, Sta97].

Using this essential property and the fact that Turing machines can be simulated by (non real-time) 2-counter automata, we proved the following result.

Theorem 4.2 ([Fin06]) *The Wadge hierarchy of the class* $\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega$ *is the Wadge hierarchy of the class* Σ_1^1 *of* ω*-languages accepted by Turing machines with a Büchi acceptance condition.*

We are going to use this result from [Fin06] to prove our main result about local ω-languages.

We first define a coding of ω-words over a finite alphabet Σ by ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ where A, B and 0 are new letters not in Σ.

We shall code an ω-word $x \in \Sigma^\omega$ by the ω-word $h(x)$ defined by

$$h(x) = A0x(1)B0^2A0^2x(2)B0^3A0^3x(3)B \cdots B0^nA0^nx(n)B \cdots$$

This coding defines a mapping $h : \Sigma^\omega \to (\Sigma \cup \{A, B, 0\})^\omega$. The function h is continuous because for all ω-words $x, y \in \Sigma^\omega$ and each positive integer n, it holds that $\delta(x, y) < 2^{-n} \to \delta(h(x), h(y)) < 2^{-n}$.

Recall that we denote $h(\Sigma^\omega)^- = (\Sigma \cup \{A, B, 0\})^\omega - h(\Sigma^\omega)$. We are going to prove that if $L(\mathcal{A}) \subseteq \Sigma^\omega$ is accepted by a real time 1-counter automaton $\mathcal{A}$ with a Büchi acceptance condition then $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$ is a local ω-language. Moreover this ω-language will have the same Wadge degree as the initial language $L(\mathcal{A})$, except for some very simple cases.

We firstly prove the following lemma.

Lemma 4.3 *Let* Σ *be a finite alphabet and* h *be the coding of* ω*-words over* Σ *defined as above. Then* $h(\Sigma^\omega)^-$ *is a local* ω*-language.*

Proof. We can easily see that $h(\Sigma^\omega)^-$ is the set of ω-words in $(\Sigma \cup \{A, B, 0\})^\omega$ which belong to one of the following ω-languages.

- $\mathcal{D}_1$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ which have not any initial segment in $A \cdot 0 \cdot \Sigma \cdot B$. It is easy to see that $\mathcal{D}_1$ is in fact a regular ω-language, hence also a local ω-language.
- $\mathcal{D}_2$ is the complement of $(A \cdot 0^+ \cdot \Sigma \cdot B \cdot 0^+)^\omega$ in $(\Sigma \cup \{A, B, 0\})^\omega$. The ω-language $(A \cdot 0^+ \cdot \Sigma \cdot B \cdot 0^+)^\omega$ is regular thus its complement $\mathcal{D}_2$ is also a regular ω-language, and thus a local ω-language.
- $\mathcal{D}_3$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ which contain a segment of the form $B \cdot 0^n \cdot A \cdot 0^m \cdot \Sigma$ for some positive integers $n \neq m$. It is easy to see (using only a unary function symbol, see for instance methods used in [Fin01]) that the finitary language containing words of the form $B.0^n.A.0^m.\Sigma$ for some positive integers $n \neq m$ is a local finitary language. Thus the ω-language $\mathcal{D}_3$ is a local ω-language.
- $\mathcal{D}_4$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ which contain a segment in $A \cdot 0^n \cdot \Sigma \cdot B \cdot 0^m \cdot A$ for some positive integers n and m with $m \neq n + 1$. Again the finitary language containing words of the form $A \cdot 0^n \cdot \Sigma \cdot B \cdot 0^m \cdot A$ with $m \neq n + 1$ is easily seen to be local. Thus the ω-language $\mathcal{D}_4$ is a local ω-language.

The class LOC_ω is closed under finite union. On the other hand it holds that $h(\Sigma^\omega)^- = \bigcup_{1 \leq i \leq 4} \mathcal{D}_i$ thus $h(\Sigma^\omega)^-$ is a local ω-language. □

We would like now to prove that if $L(\mathcal{A}) \subseteq \Sigma^\omega$ is accepted by a real time 1-counter automaton $\mathcal{A}$ with a Büchi acceptance condition then $h(L(\mathcal{A}))$ is a local ω-language. We have not been able to show this, so we are firstly going to define another ω-language $\mathcal{L}(\mathcal{A})$ which will be a local ω-language (and will be also accepted by another 1-counter Büchi automaton) and we shall prove that $h(L(\mathcal{A})) \cup h(\Sigma^\omega)^- = \mathcal{L}(\mathcal{A}) \cup h(\Sigma^\omega)^-$.

Let then $\mathcal{A}= (K, \Sigma, \Delta, q_0, F)$ be a real-time 1-counter Büchi automaton accepting the ω-language $L(\mathcal{A})\subseteq \Sigma^\omega$. The ω-language $\mathcal{L}(\mathcal{A})$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ of the form

$$Au_1v_1x_1Bw_1z_1Au_2v_2x_2Bw_2z_2A \cdots Au_nv_nx_nBw_nz_nA \cdots$$

where, for all integers $i \geq 1$, $v_i, w_i, u_i, z_i \in 0^\star$, $x_i \in \Sigma$, $|u_1| = 1$, $|u_{i+1}| = |z_i|$ and there is a sequence $(q_i)_{i\geq 0}$ of states of K and integers $j_i \in \{-1; 0; 1\}$, for $i \geq 1$, such that for all integers $i \geq 1$:

$$x_i : (q_{i-1}, |v_i|) \mapsto_{\mathcal{A}} (q_i, |v_i| + j_i)$$

and

$$|w_i| = |v_i| + j_i$$

Moreover some state $q_f \in F$ occurs infinitely often in the sequence $(q_i)_{i\geq 0}$.
Notice that the state q_0 of the sequence $(q_i)_{i\geq 0}$ is also the initial state of $\mathcal{A}$.

Lemma 4.4 *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A})\subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above. Then $\mathcal{L}(\mathcal{A})$ is a local ω-language.*

Proof. Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A})\subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above.

We first define the following finitary languages, for $q, q' \in K$.

$$T_{(q,q')} = \{vxBw \mid v, w \in 0^\star \text{ and } x \in \Sigma \text{ and } x : (q, |v|) \mapsto_{\mathcal{A}} (q', |w|)\}$$

It holds that

$$T_{(q,q')} = T^0_{(q,q')} \cup T^1_{(q,q')}$$

where

$$T^0_{(q,q')} = \{xBw \mid w \in 0^\star \text{ and } x \in \Sigma \text{ and } x : (q, 0) \mapsto_{\mathcal{A}} (q', |w|)\}$$

and

$$T^1_{(q,q')} = \{vxBw \mid v \in 0^+, w \in 0^\star, \text{ and } x \in \Sigma \text{ and } x : (q, 1) \mapsto_{\mathcal{A}} (q', 1 + \varepsilon) \text{ and } |w| = |v| + \varepsilon\}$$

We firstly notice that the finitary language $T^0_{(q,q')}$ is finite since if $xBw \in T^0_{(q,q')}$ for some $x \in \Sigma$ and $w \in 0^\star$ then $|w| = 0$ or $|w| = 1$. Thus the languages $T^0_{(q,q')}$ are local because every finite language is rational and local.

On the other hand we now recall that a linear context free language $L(G)$, over a finite alphabet Σ, is generated by a linear grammar G whose production rules are of the form: $A_i \to u_iB_iv_i$ for $1 \leq i \leq n$, and $C_i \to w_i$ for $1 \leq i \leq k$, where $\forall i \;\; u_i, v_i, w_i \in \Sigma^\star$. The variables A_i, B_i, C_i not necessarily are distinct, but are variables taken in a finite set given by G. It was proved in [Fin01] that every linear context free language is a local language.

It is now easy to see that each language $T^1_{(q,q')}$ is a finite union of linear languages and thus is a local language since the class LOC is closed under finite union. Moreover this implies also that the languages $T_{(q,q')}$, for $q, q' \in K$, are also local.

Let now X be the finite alphabet containing letters $t_{(q,q')}$, for $q, q' \in K$, and also the letters 0 and A and a letter A_0. And let $\mathcal{L} \subseteq X^\omega$ be the ω-language over X containing the ω-words of the form:

$$A_0 \cdot 0 \cdot t_{(q_0,q'_0)} \cdot A \cdot t_{(q_1,q'_1)} \cdot A \cdot t_{(q_2,q'_2)} \cdot A \cdot t_{(q_3,q'_3)} \cdots$$

where q_0 is the initial state of the automaton $\mathcal{A}$, and for each integer $i \geq 0$, $q_i, q'_i \in K$ and $q'_i = q_{i+1}$ and for which there is an accepting state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$. It is easy to see that the ω-language $\mathcal{L} \subseteq X^\omega$ is regular and thus local.

Consider now the following substitution $f : X \to \mathcal{P}((\Sigma \cup \{A, B, 0\})^\star)$ given by $t_{(q,q')} \to T_{(q,q')}$ for every $q, q' \in K$, and $A \to \{zAu \mid z, u \in 0^\star \text{ and } |z| = |u|\}$, and $A_0 \to A$.

By the definition of the ω-language $\mathcal{L}(\mathcal{A})$, it is straightforward to check that $f(\mathcal{L}) = \mathcal{L}(\mathcal{A})$. But by Theorem 2.15 the class of local ω-languages is closed under λ-free substitution by local finitary languages. Thus the ω-language $\mathcal{L}(\mathcal{A})$ is local. $\square$

We can now infer the following proposition from Lemmas 4.3 and 4.4 and from the closure of the class of local ω-languages under finite union.

Proposition 4.5 *Let $L(\mathcal{A})\subseteq \Sigma^\omega$ be an ω-language accepted by a real time 1-counter automaton $\mathcal{A}$ with a Büchi acceptance condition, and let h and $\mathcal{L}(\mathcal{A})$ be defined as above. Then $\mathcal{L}(\mathcal{A})\cup h(\Sigma^\omega)^-$ is a local ω-language.*

We now state the following lemma which will imply that $h(L(\mathcal{A}))\cup h(\Sigma^\omega)^- = \mathcal{L}(\mathcal{A})\cup h(\Sigma^\omega)^-$.

Lemma 4.6 *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A})\subseteq (\Sigma\cup\{A, B, 0\})^\omega$ be defined as above. Then $L(\mathcal{A}) = h^{-1}(\mathcal{L}(\mathcal{A}))$, i.e. $\forall x \in \Sigma^\omega \quad h(x) \in \mathcal{L}(\mathcal{A}) \longleftrightarrow x \in L(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A})\subseteq (\Sigma \cup \{A, B, 0\})^\omega$ be defined as above. Let $x \in \Sigma^\omega$ be an ω-word such that $h(x) \in \mathcal{L}(\mathcal{A})$. So $h(x)$ may be written

$$h(x) = A0x(1)B0^2A0^2x(2)B0^3A0^3x(3)B\cdots B0^nA0^nx(n)B\cdots$$

and also

$$h(x) = Au_1v_1x_1Bw_1z_1Au_2v_2x_2Bw_2z_2A\cdots Au_nv_nx_nBw_nz_nA\cdots$$

where, for all integers $i \geq 1$, $v_i, w_i, u_i, z_i \in 0^\star$, $x_i = x(i) \in \Sigma$, $|u_1| = 1$, $|u_{i+1}| = |z_i|$ and there is a sequence $(q_i)_{i\geq 0}$ of states of K and integers $j_i \in \{-1; 0; 1\}$, for $i \geq 1$, such that for all integers $i \geq 1$:

$$x_i : (q_{i-1}, |v_i|) \mapsto_{\mathcal{A}} (q_i, |v_i| + j_i)$$

and

$$|w_i| = |v_i| + j_i$$

some state $q_f \in F$ occurring infinitely often in the sequence $(q_i)_{i\geq 0}$.
In particular, $u_1 = 0$ and $u_1 \cdot v_1 = 0$ thus $|v_1| = 0$. We are going to prove by induction on the integer $i \geq 1$ that, for all integers $i \geq 1$, $|w_i| = |v_{i+1}|$. Moreover, setting $c_i = |v_i|$, we are going to prove that for each integer $i \geq 1$ it holds that

$$x_i : (q_{i-1}, c_i) \mapsto_{\mathcal{A}} (q_i, c_{i+1})$$

We have already seen that $|v_1| = 0$. By hypothesis there is a state $q_1 \in K$ and an integer $j_1 \in \{-1; 0; 1\}$ such that $x_1 : (q_0, |v_1|) \mapsto_{\mathcal{A}} (q_1, |v_1| + j_1)$, i.e. $x_1 : (q_0, 0) \mapsto_{\mathcal{A}} (q_1, j_1)$. Then $|w_1| = |v_1| + j_1 = j_1$.
We have now $|w_1 \cdot z_1| = |u_2 \cdot v_2| = 0^2$ and $|u_2| = |z_1|$ thus $|v_2| = |w_1| = j_1$. Setting $c_1 = 0$ and $c_2 = j_1 = |v_2|$, it holds that $x_1 : (q_0, c_1) \mapsto_{\mathcal{A}} (q_1, c_2)$.
Assume now that, for all integers i, $1 \leq i \leq n-1$, it holds that $|w_i| = |v_{i+1}|$ and $x_i : (q_{i-1}, c_i) \mapsto_{\mathcal{A}} (q_i, c_{i+1})$ where $c_i = |v_i|$.
We know that there is a state $q_n \in K$ and an integer $j_n \in \{-1; 0; 1\}$ such that $x_n : (q_{n-1}, |v_n|) \mapsto_{\mathcal{A}} (q_n, |v_n| + j_n)$, i.e. $x_n : (q_{n-1}, c_n) \mapsto_{\mathcal{A}} (q_n, c_n + j_n)$ and $|w_n| = |v_n| + j_n$.
On the other hand $|w_n \cdot z_n| = |u_{n+1} \cdot v_{n+1}| = 0^{n+1}$ and $|u_{n+1}| = |z_n|$ thus $|v_{n+1}| = |w_n| = c_n + j_n$. By setting $c_{n+1} = |v_{n+1}|$ we have $x_n : (q_{n-1}, c_n) \mapsto_{\mathcal{A}} (q_n, c_{n+1})$.

Finally we have proved by induction the announced claim. If for all integers $i \geq 1$, we set $c_i = |v_i|$ then it holds that

$$x_i : (q_{i-1}, c_i) \mapsto_{\mathcal{A}} (q_i, c_{i+1})$$

But there is some state $q_f \in K$ which occurs infinitely often in the sequence $(q_i)_{i\geq 1}$. This implies that $(q_{i-1}, c_i)_{i\geq 1}$ is a successful run of $\mathcal{A}$ on x thus $x \in L(\mathcal{A})$.
Conversely it is easy to see that if $x \in L(\mathcal{A})$ then $h(x) \in \mathcal{L}(\mathcal{A})$. This ends the proof of Lemma 4.6. $\square$

Remark 4.7 *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ and $\mathcal{L}(\mathcal{A})\subseteq (\Sigma\cup\{A,B,0\})^\omega$ be defined as above. Then $\mathcal{L}(\mathcal{A})$ is accepted by another real-time 1-counter Büchi automaton .*

We are now going to prove the following result.

Theorem 4.8 *The Wadge hierarchy of the class $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$ is equal to the Wadge hierarchy of the class LOC_ω.*

To prove this result we firstly consider non self dual Borel sets. We recall the definition of Wadge degrees introduced by Duparc in [Dup01] and which is a slight modification of the previous one.

Definition 4.9

(a) $d_w(\emptyset) = d_w(\emptyset^-) = 1$

(b) $d_w(L) = sup\{d_w(L') + 1 \mid L'$ non self dual and $L' <_W L\}$
(for either L self dual or not, $L >_W \emptyset$).

Wadge and Duparc used the operation of sum of sets of infinite words which has as counterpart the ordinal addition over Wadge degrees.

Definition 4.10 (Wadge, see [Wad83, Dup01]) *Assume that $X \subseteq Y$ are two finite alphabets, $Y - X$ containing at least two elements, and that $\{X_+, X_-\}$ is a partition of $Y - X$ in two non empty sets. Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, then*

$$L' + L =_{df} L \cup \{u \cdot a \cdot \beta \mid u \in X^\star,\ (a \in X_+ \textit{ and } \beta \in L') \textit{ or } (a \in X_- \textit{ and } \beta \in L'^-)\}$$

This operation is closely related to the *ordinal sum* as it is stated in the following:

Theorem 4.11 (Wadge, see [Wad83, Dup01]) *Let $X \subseteq Y$, $Y - X$ containing at least two elements, $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ be non self dual Borel sets. Then $(L + L')$ is a non self dual Borel set and $d_w(L' + L) = d_w(L') + d_w(L)$.*

A player in charge of a set $L' + L$ in a Wadge game is like a player in charge of the set L but who can, at any step of the play, erase his previous play and choose to be this time in charge of L' or of L'^-. Notice that he can do this only one time during a play.

The following lemma was proved in [Fin06]. Notice that below the emptyset is considered as an ω-language over an alphabet Γ such that $\Gamma - \Sigma$ contains at least two elements.

Lemma 4.12 *Let $L \subseteq \Sigma^\omega$ be a non self dual Borel set such that $d_w(L) \geq \omega$. Then it holds that $L \equiv_W \emptyset + L$.*

We can now prove the following lemma.

Lemma 4.13 *Let $L \subseteq \Sigma^\omega$ be a non self dual Borel set acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$. Then there is a local ω-language L' such that $L \equiv_W L'$.*

Proof. Recall first that there are regular ω-languages of every finite Wadge degree, [Sta97, Sel98]. These regular ω-languages are Boolean combinations of open sets, and they are local since every regular ω-language is local.

So we have only to consider the case of non self dual Borel sets of Wadge degrees greater than or equal to ω.

Let then $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be a non self dual Borel set, acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$, such that $d_w(L) \geq \omega$. We have seen that $h(L(\mathcal{A}))\cup h(\Sigma^\omega)^- = \mathcal{L}(\mathcal{A})\cup h(\Sigma^\omega)^-$ is a local ω-language, where the mapping $h : \Sigma^\omega \to (\Sigma \cup \{A, B, 0\})^\omega$ is defined, for $x \in \Sigma^\omega$, by:

$$h(x) = A0x(1)B0^2A0^2x(2)B0^3A0^3x(3)B \cdots B0^nA0^nx(n)B \cdots$$

We set $L' = h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$ and we now prove that $L' \equiv_W L$.

Firstly, it is easy to see that $L \leq_W L'$. In order to prove this we can consider the Wadge game $W(L, L')$. It is easy to see that Player 2 has a winning strategy in this game which consists in essentially copying the play of Player 1, except that Player 2 actually writes a beginning of the code given by h of what has been written by Player 1. This is achieved in such a way that Player 2 has written the initial word $A0x(1)B0^2A0^2x(2)B0^3A0^3x(3)B \cdots B0^nA0^nx(n)$ while Player 1 has written the initial word $x(1)x(2)x(3)x(4) \cdots x(n)$. Notice that one can admit that a player writes a finite word at each step of the play instead of a single letter. This does not change the winner of a Wadge game. At the end of a play if Player 1 has written the ω-word x then Player 2 has written $h(x)$ and thus $x \in L(\mathcal{A}) \iff h(x) \in L'$ and Player 2 wins the play.

To prove that $L' \leq_W L$, it suffices to prove that $L' \leq_W \emptyset + L$ because Lemma 4.12 states that $\emptyset + L \equiv_W L$. Consider the Wadge game $W(L', \emptyset + L)$. Player 2 has a winning strategy in this play which we now describes.

As long as Player 1 remains in the closed set $h(\Sigma^\omega)$ (this means that the word written by Player 1 is a prefix of some infinite word in $h(\Sigma^\omega)$) Player 2 essentially copies the play of player 1 except that Player 2 skips when player 1 writes a letter not in Σ. He continues forever with this strategy if the word written by player 1 is always a prefix of some ω-word of $h(\Sigma^\omega)$. Then after ω steps Player 1 has written an ω-word $h(x)$ for some $x \in \Sigma^\omega$, and Player 2 has written x. So in that case $h(x) \in L'$ iff $x \in L(\mathcal{A})$ iff $x \in \emptyset + L$.

But if at some step of the play, Player 1 "goes out of" the closed set $h(\Sigma^\omega)$ because the word he has now written is not a prefix of any ω-word of $h(\Sigma^\omega)$, then its final word will be surely outside $h(\Sigma^\omega)$ hence also inside $L' = h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$. Player 2 can now writes a letter of $\Gamma - \Sigma$ in such a way that he is now like a player in charge of the wholeset and he can now writes an ω-word u so that his final ω-word will be inside $\emptyset + L$. Thus Player 2 wins this play too.

Finally we have proved that $L \leq_W L' \leq_W L$ thus it holds that $L' \equiv_W L$. This ends the proof. □

End of Proof of Theorem 4.8.

Let $L \subseteq \Sigma^\omega$ be a Borel set acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$. If the Wadge degree of L is finite, it is well known that it is Wadge equivalent to a regular ω-language, hence also to a local ω-language. If L is non self dual and its Wadge degree is greater than or equal to ω, then we know from Lemma 4.13 that there is a local ω-language L' such that $L \equiv_W L'$.

It remains to consider the case of self dual Borel sets. The alphabet Σ being finite, a self dual Borel set L is always Wadge equivalent to a Borel set in the form $\Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2$, where (Σ_1, Σ_2) form a partition of Σ, and $L_1, L_2 \subseteq \Sigma^\omega$ are non self dual Borel sets such that $L_1 \equiv_W L_2^-$. Moreover L_1 and L_2 can be taken in the form $L_{(u_1)} = u_1 \cdot \Sigma^\omega \cap L$ and $L_{(u_2)} = u_2 \cdot \Sigma^\omega \cap L$ for some $u_1, u_2 \in \Sigma^\star$, see [Dup03]. So if $L \subseteq \Sigma^\omega$ is a self dual Borel set accepted by a real time 1-counter Büchi automaton then $L \equiv_W \Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2$, where (Σ_1, Σ_2) form a partition of Σ, and $L_1, L_2 \subseteq \Sigma^\omega$ are non self dual Borel sets accepted by real time 1-counter Büchi automata. We have already proved that there is a local ω-language L'_1 such that $L'_1 \equiv_W L_1$ and a local ω-language L'_2 such that $L_2'^- \equiv_W L_2$. Thus $L \equiv_W \Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2 \equiv_W \Sigma_1 \cdot L'_1 \cup \Sigma_2 \cdot L'_2$ and $\Sigma_1 \cdot L'_1 \cup \Sigma_2 \cdot L'_2$ is a local ω-language.

We have only considered above the Wadge hierarchy of **Borel sets**. If we assume the axiom of $\mathbf{\Sigma}^1_1$-determinacy, then Theorem 4.8 can be extended by considering the class of analytic sets instead of the class of Borel sets. In that case any set which is analytic but not Borel is $\mathbf{\Sigma}^1_1$-complete, see [Kec95], and thus there is only one more Wadge degree containing $\mathbf{\Sigma}^1_1$-complete sets. It was proved in [Fin03] that there is a $\mathbf{\Sigma}^1_1$-complete set accepted by a real-time 1-counter Büchi automaton, and it was proved in [Fin08] that there is a local ω-language which is $\mathbf{\Sigma}^1_1$-complete.

If we do not assume the axiom of $\mathbf{\Sigma}^1_1$-determinacy, we can still prove that for every $\mathbf{\Sigma}^1_1$-set L accepted by a real-time 1-counter Büchi automaton there exists a local ω-language L' such that $L \equiv_W L'$. Indeed By Lemma 4.7 of [Fin13] the conclusion of the above Lemma 4.12 is also true if L is assumed to be an analytic but non-Borel set, and thus the proof of Lemma 4.13 can be adapted if L is an analytic but non-Borel set. Notice also that in the same way the proofs of [Fin06] can be adapted to this case, using [Fin13, Lemma 4.7] instead of the above Lemma 4.12. This way we easily see that for every effective analytic but non-Borel set

$L \subseteq \Sigma^\omega$, where Σ is a finite alphabet, there exists an ω-language L' in $\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega$ such that $L' \equiv_W L$. $\square$

We can finally summarize our results by the following theorem.

Theorem 4.14 *The Wadge hierarchy of the class LOC_ω is the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega$ and also of the class Σ^1_1 of effective analytic sets.*

From this result, from the fact that for each non-null countable ordinal α the $\mathbf{\Sigma}^0_\alpha$-complete sets (respectively, the $\mathbf{\Pi}^0_\alpha$-complete sets) form a single Wadge degree, and from the results of [KMS89], we can infer the following result.

Corollary 4.15 *For each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete local ω-languages. And the supremum of the set of Borel ranks of local ω-languages is the ordinal γ^1_2, which is precisely defined in [KMS89].*

We can now also show that the topological complexity of a local ω-language may depend on the models of set theory. We first recall the following result, proved in [Fin09a]. We refer the reader to [Fin09a] and to a book on set theory like [Jec02] for more details about the notions appearing here.

Theorem 4.16 ([Fin09a]) *There is a real-time 1-counter Büchi automaton $\mathcal{A}$ which can be effectively constructed and for which the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system* **ZFC**. *Indeed it holds that :*

1. *(**ZFC + V=L**). The ω-language $L(\mathcal{A})$ is an analytic but non-Borel set.*
2. *(**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). The ω-language $L(\mathcal{A})$ is a $\mathbf{\Pi}^0_2$-set.*

Notice that, from a real time 1-counter Büchi automaton $\mathcal{A}$ reading words over the alphabet Σ, one can effectively construct a local sentence φ such that $L^\Sigma_\omega(\varphi) = h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$, where h is the mapping defined above. Moreover it follows from the previous proofs that if $L(\mathcal{A})$ is a $\mathbf{\Pi}^0_2$-set then the local ω-language $L^\Sigma_\omega(\varphi)$ is also a $\mathbf{\Pi}^0_2$-set and that if $L(\mathcal{A})$ is an analytic but non-Borel set then the local ω-language $L^\Sigma_\omega(\varphi)$ is also an analytic but non-Borel set. Thus we can now state the following result.

Theorem 4.17 *There is a local sentence φ which can be effectively constructed and a finite alphabet Σ, such that the topological complexity of the ω-language $L^\Sigma_\omega(\varphi)$ is not determined by the axiomatic system* **ZFC**. *Indeed it holds that :*

1. *(**ZFC + V=L**). The ω-language $L^\Sigma_\omega(\varphi)$ is an analytic but non-Borel set.*
2. *(**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). The ω-language $L^\Sigma_\omega(\varphi)$ is a $\mathbf{\Pi}^0_2$-set.*

As a complement we now add some high undecidability results which can be obtained from the previous constructions and from results of [Fin09b] which we now recall. As in [Fin09b], we denote below $\mathcal{A}_z$ the real time 1-counter Büchi automaton of index z reading words over a fixed finite alphabet Σ having at least two letters. We refer the reader to a textbook like [Odi89, Odi99] for more background about the analytical hierarchy of subsets of the set $\mathbb{N}$ of natural numbers.

Theorem 4.18 ([Fin09b]) *The universality, the equivalence and the inclusion problems for ω-languages accepted by real time 1-counter Büchi automata are Π^1_2-complete, i.e.:*

1. *$\{z \in \mathbb{N} \mid L(\mathcal{A}_z) = \Sigma^\omega\}$ is Π^1_2-complete*
2. *$\{(z, z') \in \mathbb{N} \mid L(\mathcal{A}_z) = L(\mathcal{A}'_z)\}$ is Π^1_2-complete*
3. *$\{(z, z') \in \mathbb{N} \mid L(\mathcal{A}_z) \subseteq L(\mathcal{A}'_z)\}$ is Π^1_2-complete*

Notice that we can associate in a recursive manner an index z to each local sentence in the recursive set $\mathbf{L}$ of local sentences given by Theorem 2.4. Then we can denote φ_z the local sentence of index z in the set $\mathbf{L}$. Using the previous constructions we can now easily show the following results.

Theorem 4.19 *The universality, the equivalence and the inclusion problems for local ω-languages are Π^1_2-complete.*

1. $\{z \in \mathbb{N} \mid L_\omega(\varphi_z) = \Gamma^\omega\}$ *is* Π^1_2*-complete*
2. $\{(z, z') \in \mathbb{N} \mid L_\omega(\varphi_z) = L_\omega(\varphi'_z)\}$ *is* Π^1_2*-complete*
3. $\{(z, z') \in \mathbb{N} \mid L_\omega(\varphi_z) \subseteq L_\omega(\varphi'_z)\}$ *is* Π^1_2*-complete*

Proof. Firstly, it is easy to see that each of these decision problems is in the class Π^1_2, using the fact that, for a local sentence φ and an ω-word $x \in \Sigma^\omega$, the sentence "$x \in L^\Sigma_\omega(\varphi)$" can be expressed by a Σ^1_1-sentence, see [Fin08, Theorem 3.6].

Secondly, we have seen that, from a real time 1-counter Büchi automaton $\mathcal{A}_z$ reading words over the alphabet Σ, one can effectively construct a local sentence φ such that $L_\omega(\varphi) = h(L(\mathcal{A})) \cup h(\Sigma^\omega)^-$, where h is the mapping defined above. Thus there is a recursive mapping $g : \mathbb{N} \to \mathbb{N}$ such that the local sentence $\varphi_{g(z)}$ is associated to $\mathcal{A}_z$, i.e. such that $L_\omega(\varphi_{g(z)}) = h(L(\mathcal{A}_z)) \cup h(\Sigma^\omega)^-$. In order to prove the completeness part of the theorem it suffices now to remark that the universality probem (respectively, the equivalence problem, the inclusion problem) for ω-languages accepted by real time 1-counter Büchi automata is reduced to the universality probem (respectively, the equivalence problem, the inclusion problem) for local ω-languages. This follows from the following equivalences, where $\Gamma = \Sigma \cup \{A, B, 0\}$ (notice that, using a standard coding, it is straighforward to prove the result for any alphabet having at least two letters):

1. $L(\mathcal{A}_z) = \Sigma^\omega \iff L^\Gamma_\omega(\varphi_{g(z)}) = \Gamma^\omega$
2. $L(\mathcal{A}_z) = L(\mathcal{A}'_z) \iff L^\Gamma_\omega(\varphi_{g(z)}) = L^\Gamma_\omega(\varphi_{g(z')})$
3. $L(\mathcal{A}_z) \subseteq L(\mathcal{A}'_z) \iff L^\Gamma_\omega(\varphi_{g(z)}) \subseteq L^\Gamma_\omega(\varphi_{g(z')})$

□

5 Concluding remarks

We have given a solution to the problem of the topological complexity of local ω-languages, by considering the Wadge hierarchy which is a great refinement of the Borel hierarchy. Local ω-languages have the same topological complexity as effective analytic sets; but they are "more effective" in the sense that the emptiness problem for local ω-languages is decidable while it is Σ^1_1-complete for ω-languages of Turing machines, see [CC89].

We have also shown that the topological complexity of a local ω-language may depend on the models of set theory. Moreover we have given the high complexity of natural decision problems for local ω-languages, like the universality, the equivalence and the inclusion problems. Notice that other problems can be shown to be Π^1_2-complete, like the cofiniteness problem, using again the previous constructions and results from [Fin09b].

Acknowledgements. We thank the anonymous referee for useful comments on a preliminary version of this paper.

References

[BS73] J. R. Büchi and D. Siefkes. Axiomatization of the monadic second order theory of ω_1. In *The monadic second order theory of all countable ordinals (Decidable theories, II)*, pages 129–217. Lecture Notes in Mathematics, Vol. 328. Springer, Berlin, 1973.

[Büc62] J.R. Büchi. On a decision method in restricted second order arithmetic. In Stanford University Press, editor, *Proceedings of the 1960 International Congress on Logic Methodology and Philosophy of Science*, pages 1–11. Stanford University Press, 1962.

[CC89] J. Castro and F. Cucker. Nondeterministic ω-computations and the analytical hierarchy. *Journal Math. Logik und Grundlagen d. Math*, 35:333–342, 1989.

[CG78] R.S. Cohen and A.Y. Gold. ω-computations on Turing machines. *Theoretical Computer Science*, 6:1–23, 1978.

[DFR01] J. Duparc, O. Finkel, and J.-P. Ressayre. Computer science and the fine structure of Borel sets. *Theoretical Computer Science*, 257(1–2):85–105, 2001.

[Dup01] J. Duparc. Wadge hierarchy and Veblen hierarchy: Part 1: Borel sets of finite rank. *Journal of Symbolic Logic*, 66(1):56–86, 2001.

[Dup03] J. Duparc. A hierarchy of deterministic context free ω-languages. *Theoretical Computer Science*, 290(3):1253–1300, 2003.

[Fin89] O. Finkel. Langages de Büchi et ω-langages locaux. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 309(20):991–994, 1989.

[Fin01] O. Finkel. Locally finite languages. *Theoretical Computer Science*, 255(1-2):223–261, 2001.

[Fin03] O. Finkel. Borel hierarchy and omega context free languages. *Theoretical Computer Science*, 290(3):1385–1405, 2003.

[Fin04] O. Finkel. Closure properties of locally finite ω-languages. *Theoretical Computer Science*, 322(1):69–84, 2004.

[Fin06] O. Finkel. Borel ranks and Wadge degrees of omega context free languages. *Mathematical Structures in Computer Science*, 16(5):813–840, 2006.

[Fin08] O. Finkel. Topological complexity of locally finite ω-languages. *Archive for Mathematical Logic*, 47(6):625–651, 2008.

[Fin09a] O. Finkel. The complexity of infinite computations in models of set theory. *Logical Methods in Computer Science*, 5(4:4):1–19, 2009.

[Fin09b] O. Finkel. Highly undecidable problems for infinite computations. *RAIRO-Theoretical Informatics and Applications*, 43(2):339–364, 2009.

[Fin13] O. Finkel. The determinacy of context-free games. *The Journal of Symbolic Logic*, 78(4):1115–1134, 2013. Preprint available from http://arxiv.org/abs/1312.3412.

[FR96] O. Finkel and J. P. Ressayre. Stretchings. *The Journal of Symbolic Logic*, 61(2):563–585, 1996.

[Jec02] T. Jech. *Set theory, third edition*. Springer, 2002.

[Kec95] A. S. Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.

[KMS89] A. S. Kechris, D. Marker, and R. L. Sami. Π_1^1 Borel sets. *Journal of Symbolic Logic*, 54(3):915–920, 1989.

[LT94] H. Lescow and W. Thomas. Logical specifications of infinite computations. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *A Decade of Concurrency*, volume 803 of *Lecture Notes in Computer Science*, pages 583–621. Springer, 1994.

[Mey75] Albert R. Meyer. Weak monadic second order theory of succesor is not elementary-recursive. In *Logic Colloquium (Boston, Mass., 1972–1973)*, volume 453 of *Lecture Notes in Mathematics*, pages 132–154. Springer, Berlin, 1975.

[Mos80] Y. N. Moschovakis. *Descriptive set theory*. North-Holland Publishing Co., Amsterdam, 1980.

[Odi89] P.G. Odifreddi. *Classical Recursion Theory, Vol I*, volume 125 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1989.

[Odi99] P.G. Odifreddi. *Classical Recursion Theory, Vol II*, volume 143 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[Pin96] J.-E. Pin. Logic, semigroups and automata on words. *Annals of Mathematics and Artificial Intelligence*, 16:343–384, 1996.

[PP04] D. Perrin and J.-E. Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.

[Res88] J. P. Ressayre. Formal languages defined by the underlying structure of their words. *The Journal of Symbolic Logic*, 53(4), 1988.

[Sel98] V.L. Selivanov. Fine hierarchy of regular ω-languages,. *Theoretical Computer Science*, 191:37–59, 1998.

[Sim92] P. Simonnet. *Automates et théorie descriptive*. PhD thesis, Université Paris VII, 1992.

[Sta97] L. Staiger. ω-languages. In *Handbook of formal languages, Vol. 3*, pages 339–387. Springer, Berlin, 1997.

[Tho90] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, Formal models and semantics, pages 135–191. Elsevier, 1990.

[Tho97] W. Thomas. Languages, automata, and logic. In *Handbook of formal languages, Vol. 3*, pages 389–455. Springer, Berlin, 1997.

[Wad83] W. Wadge. *Reducibility and determinateness in the Baire space*. PhD thesis, University of California, Berkeley, 1983.